THE LITTLEWOOD–OFFORD PROBLEM AND INVERTIBILITY OF RANDOM MATRICES

Chan Pak Hop

Bachelor of Science in Actuarial Science

Name: Chan Pak Hop
University No.: 3035567356
Student's Major:
Bachelor of Science in Actuarial Science

Research Colloquium for
Science UG Students 2021-22

Abstract

We study the paper *The Littlewood–Offord problem and invertibility of random matrices* by Mark Rudelson and Roman Vershynin in 2008. By considering different types of vectors, through applying some inequalities and tools in probability theory and random matrix theory, we find a tail probability for smallest singular value, which can be transformed to the invertibility probability for random matrices.

Introduction

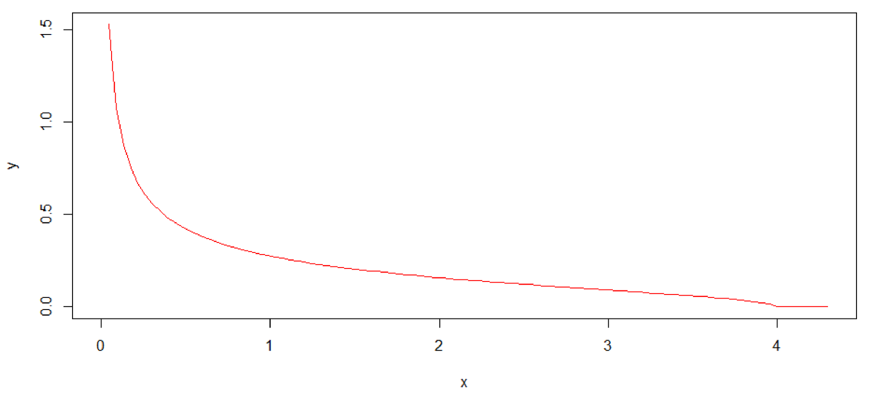
Random matrices are square or rectangular matrices with at least some of the entries as random variables. It is useful in:

- Physics
  e.g. Wigner: energy levels for heavy atoms
- Number theory
  e.g. zeroes of Riemann zeta function
- Data science/Machine learning

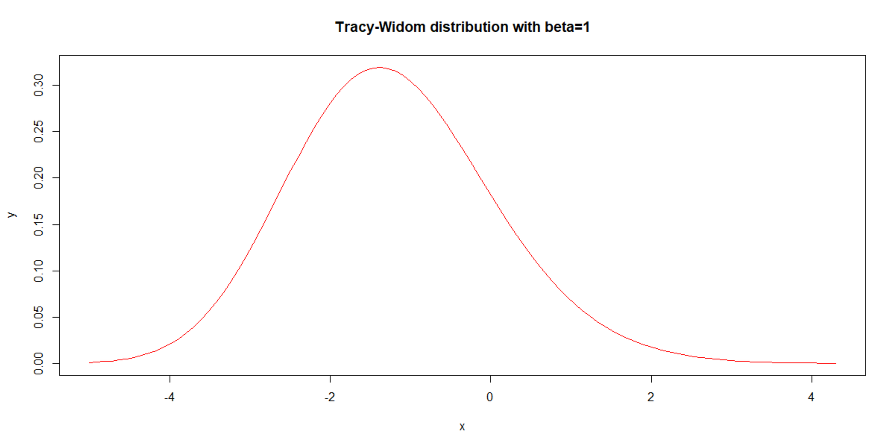
We would like to study invertibility of random matrices. This can be done by considering the *singular values*, the square roots of non-negative eigenvalues of self-adjoint operator A^*A.
The matrix is invertible if the smallest singular value, $s_n(A)$, is greater than zero.

Facts and assumptions

Global behaviours of singular values are described by *Marchenko-Pastur law*.



The largest singular value is described by the *Tracy-Widom law*.



The smallest singular value is the infimum of $||Ax||_2$ for x in the n-sphere. We work with $n \times n$ square matrices A, where entries are:

- independent
- real
- with variances at least 1
- subgaussian

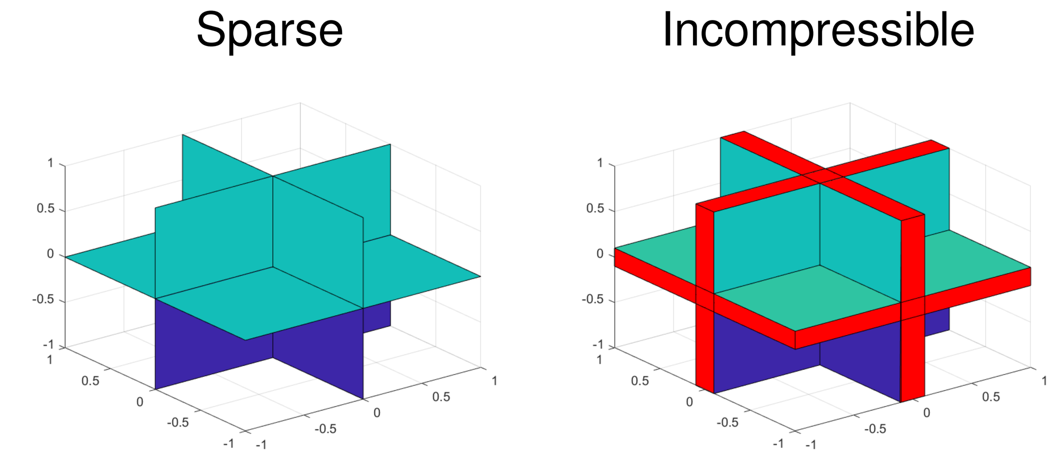
We want to find an upper bound for

$$\mathbb{P}\left(s_n(A) \leq \varepsilon n^{-1/2}\right).$$

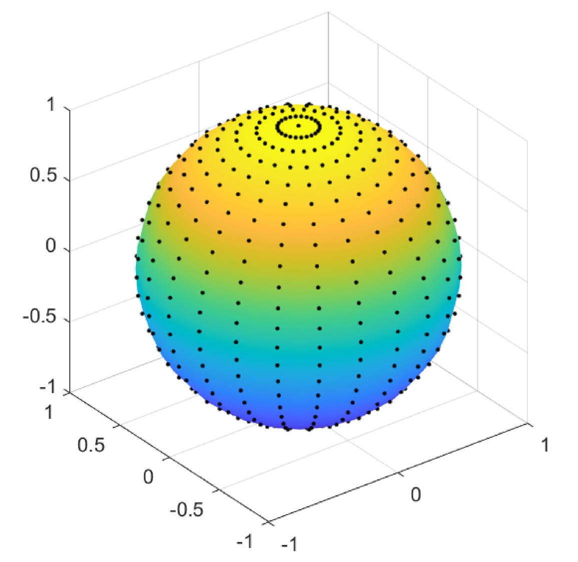
So if we put $\varepsilon = 0$, gives invertibility probability bound.

Key steps

Vectors in the $(n-1)$-sphere are classified as sparse, compressible and incompressible vectors.



We make use of a union bound argument, which means probability of union of events is at most sum of probability of individual events.
Since the union bound argument must be applied to countable events, we apply it to an ε-net of the sphere, which contains countable vectors with desirable properties that represent the whole sphere.

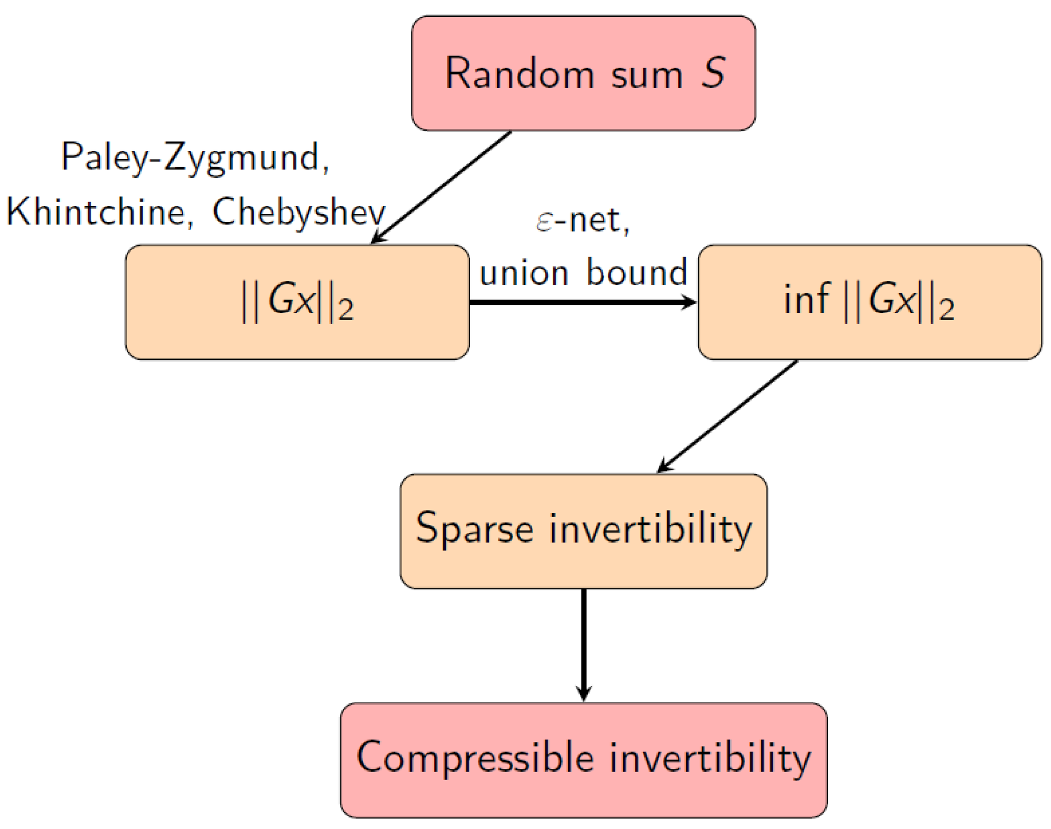


$||Ax||_2$ is related to *random sums* $S = \sum_{k=1}^{n} a_k \xi_k$, by considering the coordinates of vector x as coefficients for random variables in the matrix A. Thus, in fact by bounding for the norm of Ax we are actually bounding the random sum S.

Compressible vectors

We start with sparse vectors because many coordinates of sparse vectors of zeroes. This means we can ignore many columns in the random matrices, which allows us to use an ε-net effectively to get the invertibility for sparse vectors.
Then, we make use of the fact that compressible vectors are close to sparse vectors to obtain the probability for sparse vectors.

Outline of steps to obtain tail probability for compressible vectors



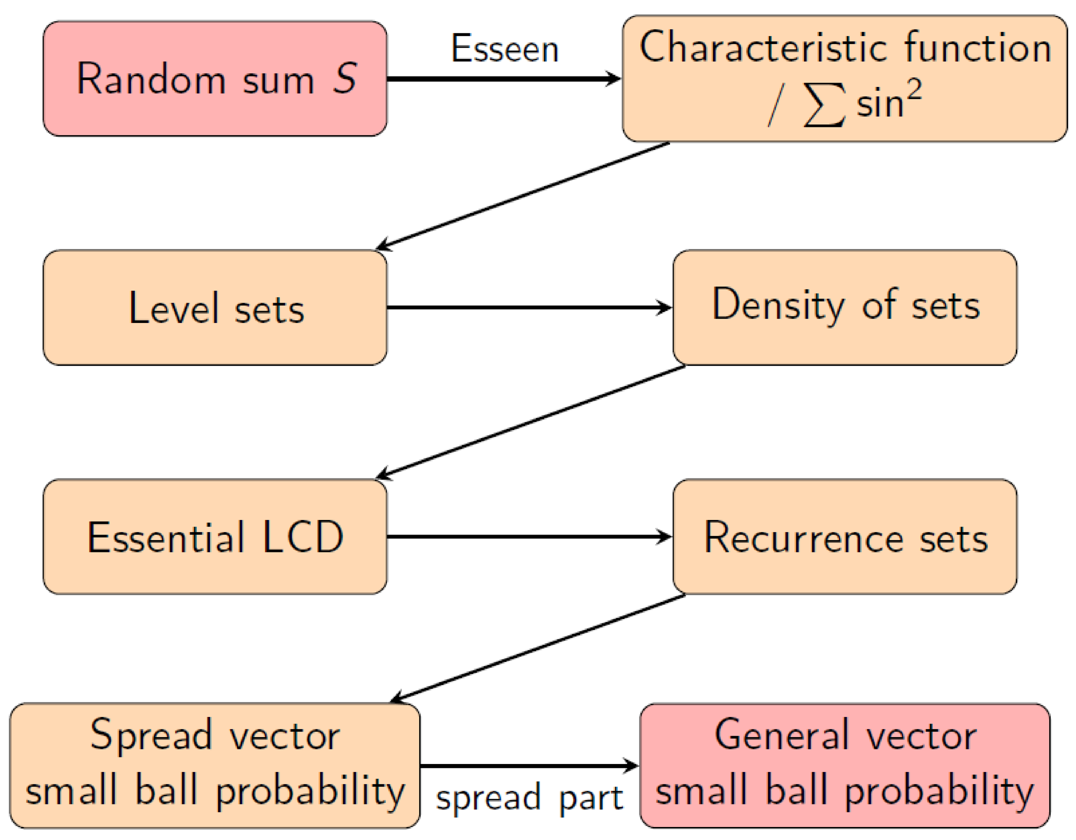
Incompressible vectors

We need to estimate **small ball probability**, which is related to the **Littlewood-Offord problem**.

$$p_\varepsilon(a) := \sup_{v \in \mathbb{R}} \mathbb{P}\left(|S - v| \leq \varepsilon\right)$$

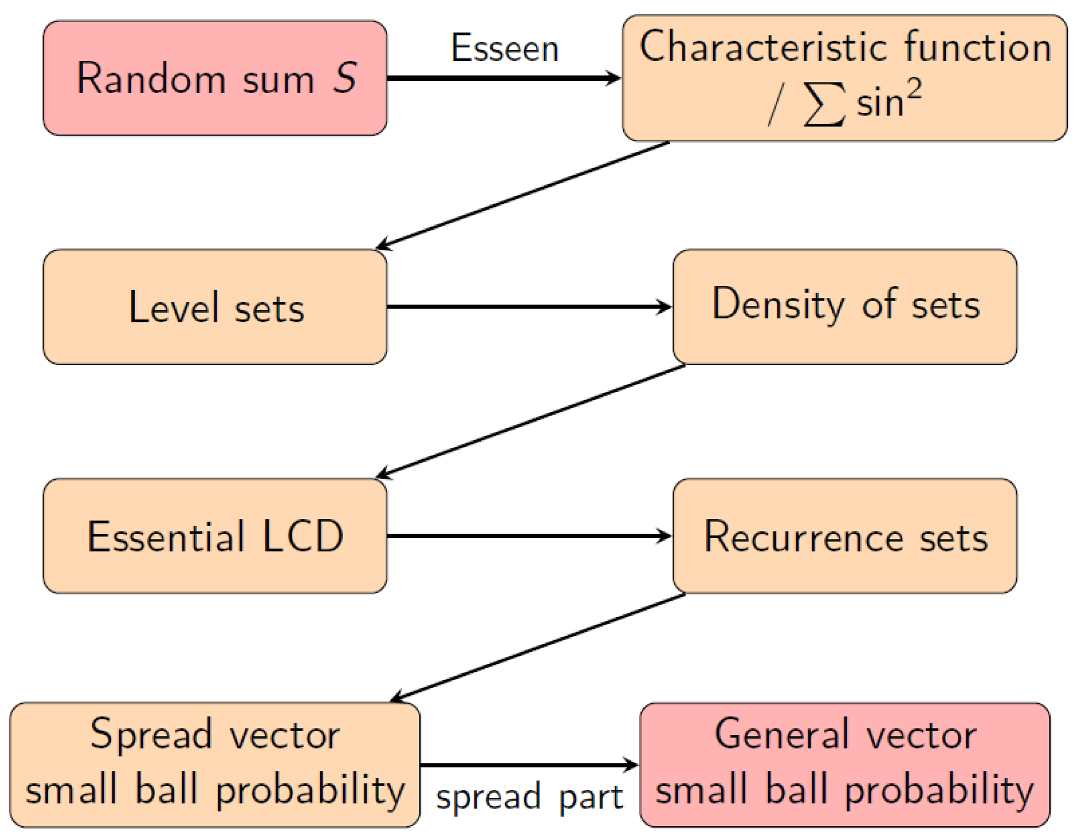
This is a measure of how "spread" S is. We would like to obtain the small ball probability for general vectors, which will then be used to establish the invertibility probability for incompressible vectors.

Outline of steps to obtain small ball probability for general vectors



Then, we try to consider the distance between an individual column vector and the span of other column vectors. If they are close, the smallest singular value of the matrix will also be small. The span of other column vectors can be summarised with the **random normal** of the span.

Outline of steps to obtain tail probability for incompressible vectors



Conclusions

Let A be an $n \times n$ matrix. Within each row, entries are real, independent, centered, with variances of at least 1 and subgaussian moments bounded by B. All rows are independent and follow the same distribution. Then, for any $\varepsilon \geq 0$,

$$\mathbb{P}\left(s_n(A) \leq \varepsilon n^{-1/2}\right) \leq C\varepsilon + c^n,$$

where $C > 0$ and $c \in (0, 1)$ depends only on B.
Thus, if the assumptions are met, the singular probability for random matrices are exponentially small.

References

[1] Aurélien Garivier, Walid Hachem, Adrien Hardy, and Jamal Najim. A survey on the eigenvalues local behavior of large complex correlated wishart matrices. *ESAIM. Proceedings and surveys*, 51:150–174, 2015.
[2] Sofia Jijon. Simple poster (portrait). `https://github.com/sjijon/TeX-templates/tree/main/Tikzposter%20posters/Simple%20poster`, 2021.
[3] Iain M Johnstone. On the distribution of the largest eigenvalue in principal components analysis. *The Annals of statistics*, 29(2):295–327, 2001.
[4] Iain M. Johnstone, Zongming Ma, Patrick O. Perry, and Morteza Shahram. *RMTstat: Distributions, Statistics and Tests derived from Random Matrix Theory*, 2014. R package version 0.3.
[5] A.E Litvak, A Pajor, M Rudelson, and N Tomczak-Jaegermann. Smallest singular value of random matrices and geometry of random polytopes. *Advances in mathematics (New York. 1965)*, 195(2):491–523, 2005.
[6] V A Marčenko and L A Pastur. Distribution of eigenvalues for some sets of random matrices. *Mathematics of the USSR. Sbornik*, 1(4):457–483, 1967.
[7] Mark Rudelson and Roman Vershynin. The littlewood–offord problem and invertibility of random matrices. *Advances in mathematics (New York. 1965)*, 218(2):600–633, 2008.
[8] Terence Tao. *Topics in random matrix theory.* Graduate studies in mathematics ; v. 132. American Mathematical Society, Providence, R.I., 2012.
[9] C. A Tracy and H Widom. Level-spacing distributions and the airy kernel. *Communications in mathematical physics*, 159(1):151–174, 1994.
[10] John von Neumann and H. H Goldstine. Numerical inverting of matrices of high order. *Bulletin (new series) of the American Mathematical Society*, 53(11):1021–1099, 1947.